Assignment-3

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Question-1 Consider the following problems (18 marks)

(*a*) $max_{x,y}\ x^2+y^2$ subject to $r^2 \le 2x^2+6y^2 \le s^2$ with $0<r<s$

(*b*) $\min_{x,y}\ x^2+y^2$ subject to $r^2 \le 2x^2+6y^2 \le s^2$ with $0<r<s$

(i) Solve problem (*a*)

$$L = max_{x,y}\ x^2+y^2-\lambda_1(2x^2+6y^2-s^2)+\lambda_2(2x^2+6y^2-r^2)$$

(1) $\dfrac{\partial L}{\partial x}=2x^*-4\lambda_1^*x+4\lambda_2^*x=0$

(2) $\dfrac{\partial L}{\partial y}=2y^*-12\lambda_1^*y^*+12\lambda_2^*y^*=0$

(3) $\lambda_1^*(2x^{*2}+6y^{*2}-s^2)\ge 0\ \&\ \lambda_1^*\ge 0$

(4) $\lambda_2^*(-2x^{*2}-6y^{*2}+r^2)\ge 0\ \&\ \lambda_2^*\ge 0$

$\lambda_1^*>0\ \&\ \lambda_2^*>0$ not possible because $2x^{*2}+6y^{*2}=s^2$ and $2x^{*2}+6y^{*2}=r^2$ cannot both be true

$\lambda_1^*\ \&\ \lambda_2^*=0 \xrightarrow{(1),(2)} x^*,y^*=0 \Rightarrow$ contradicts $2x^{*2}+6y^{*2}-r^2>0$

$Let\lambda_1^*=0 \xrightarrow{(1),(2)} \lambda_2^*<0 \rightarrow$ not acceptable

$$Let\lambda_1^*>0\ \&\ \lambda_2^*=0 \xrightarrow{(1),(2)} \begin{cases} x^*=0\ \&\ \lambda_1^*=1/6 \xrightarrow{from} 6y^{*2}-s^2=0 \Rightarrow y^*=\pm s/\sqrt{6} \rightarrow f^*=s^2/6 \\ y^*=0\ \&\ \lambda_1^*=1/2 \xrightarrow{from} 2x^{*2}-s^2=0 \Rightarrow x=\pm s/\sqrt{2} \rightarrow f^*=s^2/2 \end{cases}$$

(ii) Solve problem (*b*)

This problem can be rewritten as

$$f^* = -max_{x,y} -x^2-y^2 \text{ subject to } r^2 \le 2x^2+6y^2 \le s^2 \text{ with } 0<r<s$$

Lagrangian

$$L=-x^2-y^2-\lambda_1(2x^2+6y^2-s^2)+\lambda_2(2x^2+6y^2-r^2)$$

(1) $\dfrac{\partial L}{\partial x}=-2x^*-4\lambda_1^*x^*+4\lambda_2x^*=0$

(2) $\dfrac{\partial L}{\partial y}=-2y^*-12\lambda_1^*y^*+12\lambda_2^*y^*=0$

(3) $\lambda_1^*(2x^{*2}+6y^{*2}-s^2)\ge 0\ \&\ \lambda_1^*\ge 0$

(4) $\lambda_2(2x^{*2}+6y^{*2}-r^2)\ge 0\ \&\ \lambda_2^*\ge 0$

$\lambda_1^* > 0 \,\&\, \lambda_2^* > 0$ not possible because $2x^{*2} + 6y^{*2} = s^2$ and $2x^{*2} + 6y^{*2} = r^2$ both cannot be true

$\lambda_1^* \,\&\, \lambda_2^* = 0 \xrightarrow{(1),(2)} x^*, y^* = 0 \Rightarrow$ contradicts $2x^{*2} + 6y^{*2} - r^2 > 0$

$Let \lambda_2^* = 0 \xrightarrow{(1),(2)} \lambda_1^* < 0 \rightarrow$ not acceptable

$$Let \lambda_2^* > 0 \,\&\, \lambda_1^* = 0 \xrightarrow{(1),(2)} \begin{cases} x^* = 0 \,\&\, \lambda_2^* = 1/6 \xrightarrow{from} 6y^{*2} - r^2 = 0 \Rightarrow y^* = \pm s/\sqrt{6} \rightarrow f^* = r^2/6 \\ y^* = 0 \,\&\, \lambda_1^* = 1/2 \xrightarrow{from} 2x^{*2} - r^2 = 0 \Rightarrow x^* = \pm s/\sqrt{2} \rightarrow f^* = r^2/2 \end{cases}$$

(iii) How much does the optimal value of the function change if s changes by .1 unit in problem (a). How much does the optimal value of the function change if r changes by .1 unit in problem (a).

We use envelope theorem to answer this question

$$L = x^2 + y^2 - \lambda_1(2x^2 + 6y^2 - s^2)$$

$$\Delta f^* = \frac{\partial L}{\partial s}\Delta s = \frac{\partial L}{\partial s}\Delta s = 2s\lambda_1 \Delta s = 2s = 2s(1/2)\times 0.1 = 0.1s$$

This can also be calculated directly by taking differential from $f^* = s^2/2$

Since r is not binding its change won't affect the optimal value of the objective function.

(iv) Check the second order condition for problem (b).

Let $L = -x^2 - y^2 + \lambda_2(2x^2 + 6y^2 - r^2)$

$$\bar{H} = \begin{pmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 4x^* & 12y^* \\ 4x^* & -2 + 4\lambda_2^* & 0 \\ 12y^* & 0 & -2 + 12\lambda_2^* \end{pmatrix}$$

$$|\bar{H}| = -4x^* \begin{vmatrix} 4x^* & 0 \\ 12y^* & -2 + 12\lambda_2 \end{vmatrix} + 12y \begin{vmatrix} 4x^* & -2 + 4\lambda_2 \\ 12y^* & 0 \end{vmatrix}$$

$x^* = 0; \lambda_2^* = 1/6; y^* = \pm r/\sqrt{6} \Rightarrow |\bar{H}| = 12y^*[0 - (-2 + 4\lambda_2^*)12y] = 144y^{*2}[-(-2 + 4\lambda_2^*)] > 0 \Rightarrow \max$

$y^* = 0; \lambda_2^* = 1/2; x^* = \pm r/\sqrt{2}) \Rightarrow |\bar{H}| = -4x^*[4x^*(-2 + 12\lambda_2^*) - 0] = -4x^{*2}(-2 + 12\lambda_2^*) < 0 \Rightarrow \min$

(v) What are the geometric interpretations of (a) and (b)?

The admissible set is the area between two ellipses and the problem (a) and (b) are equivalent to finding the largest and smallest distance from the origin to a point in this admissible set.

Question-2 Find the solution to (10 marks)

$$\min_{\mathbf{x}} - \sum_{i=1}^{N} \log(\alpha_i + x_i) \text{ subject to } x_i \geq 0 \text{ and } \sum_{i=1}^{N} x_i = 1 \text{ with } \alpha_i > 0$$

Form the Lagrangian $L = -\sum_{i=1}^{N} \log(\alpha_i + x_i) - \mu\left(\sum_{i=1}^{N} x_i - 1\right) - \lambda_i x_i$ and first order condition

$$\begin{cases} (1)\ L_{xi} = \dfrac{-1}{\alpha_i + x_i^*} - \mu^* - \lambda_i^* = 0 \\ (2)\ L_\mu = \sum_{i=1}^{N} x_i^* - 1 = 0 \\ (3)\ \lambda_i^* = 0 \text{ if } x_i > 0\ \&\ \lambda_i^* > 0 \text{ if } x_i^* = 0 \end{cases}$$

Then

$$\left.\begin{aligned} &if\ \lambda_i^* = 0\ \Rightarrow \frac{-1}{\alpha_i + x_i^*} - \mu^* = 0 \Rightarrow x_i = -1/\mu^* - \alpha_i > 0 \\ &if\ \lambda_i^* > 0 \Rightarrow x_i^* = 0 \end{aligned}\right\} \Rightarrow x_i^* = \max\{-1/\mu^* - \alpha_i, 0\}$$

μ^* can be obtained using $\sum_{i=1}^{N} x_i^* = \sum_{i=1}^{N} \max\{-1/\mu^* - \alpha_i, 0\} = 1$

So to solve this problem, we first need to solve $\sum_{i=1}^{N} \max\{-1/\mu^* - \alpha_i, 0\} = 1$ to obtain μ^* and then find $x_i^* = \max\{-1/\mu - \alpha_i, 0\}$

Is the objective function $-\sum_{i=1}^{N} \log(\alpha_i + x_i)$ concave or convex? Prove your answer.

Let $f = -\sum_{i=1}^{N} \log(\alpha_i + x_i)$ then the first and second order partial derivatives and Hessian are

$$f_i = -\frac{1}{x_i + \alpha_i} \Rightarrow \begin{cases} f_{ii} = \dfrac{1}{(x_i + \alpha_i)^2} \\ f_{ij} = 0 \text{ if } i \neq j \end{cases} \Rightarrow H = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{(x_1 + \alpha_1)^2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \dfrac{1}{(x_n + \alpha_n)^2} \end{pmatrix}$$

This is obviously positive definite $\Rightarrow$ Function is Convex

Question-3 (10 marks)

Suppose a consumer has a wealth of W. There is a probability p of a loss of L if an adverse event happens. The consumer can buy insurance that will pay him Q in case that the loss happens. The consumer has to pay π per dollar insured as the premium. The consumer's problem can be formulated as

$$\max_Q\ pU(W - L - \pi Q + Q) + (1 - p)U(W - \pi Q)$$

i) Find the first order condition.

$$p(1-\pi)U'(W-L-\pi Q^*+Q^*)-\pi(1-p)U'(W-\pi Q^*)=0$$

ii) Note that the expected profit for the insurance company is $(1-p)\pi Q - p(1-\pi)Q$. Suppose that the market is competitive which forces the expected profit to be zero. In this case, find π.

$(1-p)\pi Q - p(1-\pi)Q = 0 \Rightarrow (1-p)\pi = p(1-\pi) \Rightarrow \pi = p$

iii) If the consumer is strictly risk-averse i.e. $d^2U/dW^2 < 0$, show that under (ii) the consumer fully insure against the loss i.e. $Q^* = L$

$$p(1-\pi)U'(W-L-\pi Q^*+Q^*)-\pi(1-p)U'(W-\pi Q^*)=0$$
$$\Rightarrow U'(W-L-\pi Q^*+Q^*)-U'(W-\pi Q^*)=0$$
$$\Rightarrow W-L-\pi Q^*+Q^* = W-\pi Q^*$$
$$\Rightarrow Q^* = L$$

Question-4 (12 marks)

An investor must choose a portfolio $\mathbf{x}=(x_1,....,x_n)^T$ where x_j is the proportion of assets invested in j-th security. The return to the security is $M=\boldsymbol{\mu}\mathbf{x}=\sum_{j=1}^{n}\mu_j x_j$ where $\boldsymbol{\mu}$ is the vector containing mean returns to each security. The risk on the portfolio is measured by the variance of returns $V=\mathbf{x}^T\boldsymbol{\Sigma}\,\mathbf{x}=\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{jk}x_j x_k$ where $\boldsymbol{\Sigma}$ is the variance-covariance matrix of security returns. A portfolio is efficient if there is no other portfolio with either a higher return and lower risk or with a lower risk at the same level of return.

1. For the problem of

$$\max_x M(\mathbf{x}) \text{ subject to } \mathrm{V}(\mathbf{x}) \le V_0, \mathbf{x} \ge \mathbf{0}, \mathbf{i}^T\mathbf{x}=1$$

find the first order conditions and show the solution yields an efficient portfolio.

$$\max_x M(\mathbf{x}) \text{ subject to } \mathrm{V}(\mathbf{x}) \le V_0, \mathbf{x} \ge \mathbf{0}, \mathbf{i}^T\mathbf{x}=1$$

$$L=\boldsymbol{\mu}^T\mathbf{x}-\delta\left(\mathbf{i}^T\mathbf{x}-1\right)-\gamma(\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}-V_0)+\boldsymbol{\lambda}^T\mathbf{x}$$

First order conditions

$$\begin{cases}\boldsymbol{\mu}^T-\delta^*\mathbf{i}^T-2\gamma^*\mathbf{x}^{*T}\boldsymbol{\Sigma}+\boldsymbol{\lambda}^{*T}=\mathbf{0}\xrightarrow{\text{Transposing}}\gamma^*\mathbf{x}^*=\frac{1}{2}\boldsymbol{\Sigma}^{-1}\left(-\boldsymbol{\mu}+\delta^*\mathbf{i}-\boldsymbol{\lambda}^*\right)\\ \mathbf{i}^T\mathbf{x}^*=1\\ \gamma^*(\mathbf{x}^{*T}\boldsymbol{\Sigma}\mathbf{x}^*-V_0)=0 \;\&\; \gamma^*\ge 0\\ \boldsymbol{\lambda}^*\odot\mathbf{x}^*\ge\mathbf{0} \;\&\; \boldsymbol{\lambda}^*\ge\mathbf{0} \text{ where } \odot \text{ means element by element multiplication i.e. } \{\lambda_i^* x_i^*>0\}\end{cases}$$

Suppose $\mathbf{x}^*$ *is not efficient then there should be another* $\mathbf{x}^{**}$ *that obtains a higher return with a variance less than or equal to* V_0 *but this contradicts* $\mathbf{x}^*$ *being the point of maximum subject to* $\mathrm{V}(\mathbf{x}) \le V$.

2. For the problem of

$$\min_x V(\mathbf{x}) \text{ subject to } \mathrm{M}(\mathbf{x}) \ge M_0, \mathbf{x} \ge \mathbf{0}, \mathbf{i}^T\mathbf{x} = 1$$

find the first order conditions and show the solution yields an efficient portfolio.

$$L = \mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} - \delta\left(\mathbf{i}^T\mathbf{x} - 1\right) - \gamma(\boldsymbol{\mu}^T\mathbf{x} - M_0) - \boldsymbol{\lambda}^T\mathbf{x}$$

$$\begin{cases} 2\mathbf{x}^T\boldsymbol{\Sigma}^* - \delta^*\mathbf{i}^T - \gamma^*\boldsymbol{\mu}^T - \boldsymbol{\lambda}^{*T} = \mathbf{0} \xrightarrow{\text{Transposing}} \mathbf{x}^* = \frac{1}{2}\boldsymbol{\Sigma}^{-1}\left(\delta^*\mathbf{i} + \gamma^*\boldsymbol{\mu} + \boldsymbol{\lambda}^*\right) \\ \mathbf{i}^T\mathbf{x}^* = 1 \\ \gamma^*(\boldsymbol{\mu}^T\mathbf{x}^* - M_0) \ge 0 \ \& \ \gamma^* \ge 0 \\ \boldsymbol{\lambda}^* \odot \mathbf{x}^* \ge \mathbf{0} \ \& \ \boldsymbol{\lambda}^* \ge \mathbf{0} \end{cases}$$

Suppose $\mathbf{x}^*$ *is not efficient then there should be another* $\mathbf{x}^{**}$ *that obtains a lower risk with a a return more than or equal to* μ_0 *but this contradicts* $\mathbf{x}^*$ *being the point of minimum subject to* $\mathrm{M}(\mathbf{x}) \ge M_0$.