Econ 2125 Optimisation for Economics and Financial Economics *

Week 5 Tutorial Solutions

Question 1 Prove that a closed subset of a compact set in $\mathbb{R}^m$ is compact.

Proof: Consider a compact set A and a closed subset $C \subset A$.

Since A is compact, it is bounded. That is, there exists $r > 0$ such that $A \subset B_0(r)$.

Since $C \subset A$, we know $C \subset B_0(r)$. Thus C is bounded.

Since C is also closed, we know C is compact.

Question 2

(a) Prove that the intersection of compact sets is compact.

Proof: Let S be the intersection of compact sets S_i: $S = \cap_i S_i$. (There can be finitely or infinitely many S_i).

Pick any S_i. Since S_i is compact, we know it is bounded. Since $S \subset S_i$, we know from Question 1 that S is bounded.

Since each S_i is compact, and thus closed, from Theorem 12.10 we know that $S = \cap_i S_i$ is closed.

Hence S is compact.

(b) Prove that the finite union of compact sets is compact.

Let S be the union of finitely many compact sets $S = \cup_{i=1}^n S_i$. Since each S_i is compact, and thus bounded, for each i there is a $B_i \geq 0$ such that for $x_i \in S_i$, $||x_i|| \leq B_i$.

Denote $B = \max\{B_1, ..., B_n\}$. For each $x \in S$, we know x belongs to some S_i. Thus $||x|| \leq B$. So S is bounded.

Since S_i is compact, and thus closed, by Theorem 12.10, we know $S = \cup_{i=1}^n S_i$ is closed. Hence S is compact.

(c) Is the following statement true or false? If it is true, prove it. If it is false, present a counter example:

Statement: *An infinite union of compact sets must be compact.*

Answer: The statement is false. Consider the following counter example: Let $S_n = [-\frac{n}{n+1}, \frac{n}{n+1}]$. Then $\cup_{n=1}^{\infty} S_n = (-1, 1)$, which is not compact.

Question 3: Draw a level curve for each of the following functions:

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(a) $f(x,y) = y - 2x$
Answer: Given a constant c, $y - 2x = c$ implies $y = 2x + c$.

(b) $f(x,y) = y/x$
Answer: Given a constant c, $\frac{y}{x} = c$ implies $y = cx$, $x \neq 0$.

(c) $f(x,y) = y - x^2$
Answer: Given a constant c, $y - x^2 = c$ implies $y = c + x^2$.

Question 4: Write the following linear functions in matrix form:
(a) $f(x,y,z) = 2x - 3y + 5z$
Answer:

$$[2 \ -3 \ 5] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) $f(x,y) = (2x - 3y, x - 4y, x)$
Answer:

$$\begin{bmatrix} 2 & -3 \\ 1 & -4 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) $f(x,y,z) = (x - z, 2x + 3y - 6z, x + 2y)$
Answer:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -6 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Question 5: Write the following quadratic forms in matrix form:
(a) $x^2 - 2xy + y^2$
Answer:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) $5x^2 - 10xy - y^2$
Answer:
$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -5 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) $x^2 + 2y^2 + 3z^2 + 4xy - 6xz + 8yz$
Answer:
$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Question 6: Write two functions from $\mathbb{R}$ to $\mathbb{R}$ that are not polynomials.
Answer:
$f(x) = \ln(x^2 + 1)$
$f(x) = 2^x$

Question 7
Consider the function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} x^2 \text{ if } x \in \mathbb{Q} \\ 2x - 1 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Is f continuous at $x = 1$? Explain.
Answer:
f is continuous at $x = 1$.
Consider any sequence $\{x_n\}$ such that $x_n \to 1$.
For any $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{3}\}$.
Since $x_n \to 1$, there exists N such that for $n \geq N$, $|x_n - 1| < \delta$. Thus for $n \geq N$, we have
(i) If $x_n \in Q$, then

$$\begin{aligned} & |f(x_n) - f(1)| \\ = & |x_n^2 - 1| \\ = & |x_n + 1||x_n - 1| \\ < & 3 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(ii) If $x_n \notin Q$, then

$$\begin{aligned} & |f(x_n) - f(1)| \\ = & |2x_n - 1 - 1| \\ = & 2|x_n - 1| \\ < & 2 \cdot \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Thus for $n \geq N$, $|f(x_n) - f(1)| < \varepsilon$. So $f(x_n) \to f(1)$. Hence f is continuous at 1.

Question 8: For each of the following functions, what is its domain and range? Is it one-to-one? If it is one-to-one, write the expression for the inverse function. Is it onto?

(a) $f(x) = 3x - 7$
Answer:
Domain: $\mathbb{R}$
Target space: $\mathbb{R}$
Range: $\mathbb{R}$
The function is one-on-one and onto.
Inverse function: Let $y = 3x - 7$, then $x = \frac{y}{3} + \frac{7}{3}$. So $f^{-1}(y) = \frac{y}{3} + \frac{7}{3}$.

(b) $f(x) = e^x$
Answer:
Domain: $\mathbb{R}$
Target space: $\mathbb{R}$
Range: $\mathbb{R}_{++}$
The function is one-on-one, but not onto on its target space.
Inverse function: Let $y = e^x$. Then $x = \ln y$. So $f^{-1}(y) = \ln y$.

(c) $f(x) = \sqrt{x-1}$
Answer:
Domain: $[1, +\infty)$.
Target space: $\mathbb{R}$
Range: $\mathbb{R}_+$
The function is one-on-one, but not onto on its target space.
Inverse function: Let $y = \sqrt{x-1}$. Note that this means it must be $y \geq 0$. Then $y^2 = x - 1 \Rightarrow x = y^2 + 1$. So $f^{-1}(y) = y^2 + 1$, $y \geq 0$.

(d) $f(x) = x^2 - 1$
Answer:
Domain: $\mathbb{R}$.
Target space: $\mathbb{R}$.
Range: $[-1, +\infty)$.
The function is not one-on-one, and not onto on its target space.