THE AUSTRALIAN NATIONAL UNIVERSITY

Mid Semester Examination

Semester One, 2015

Optimisation for Economics and Financial Economics

and

Mathematical Techniques in Economics I

(ECON2125/4021/8013)

Writing Period: 2 hours

Study Period: 15 minutes

Permitted Materials: None

All questions to be completed in the script book provided

INSTRUCTIONS

- Read the questions carefully.
- There are 16 questions. Questions 1–14 are worth 2 marks. Question 15 is worth 4 marks. Question 16 is worth 8 marks.
- To maximize your marks, explain the steps in your arguments while at the same time avoiding irrelevant discussions. Try to be clear and succinct.
- In solving the questions, you are welcome to use any fact that you remember from the lecture slides without any form of proof. However, you should clearly state the relevant fact.
- You do not need to do the questions in order, as long as you clearly mark in your answer sheet which question you are addressing.

QUESTIONS

Question 1. The *mode* of a density p on $\mathbb{R}$ is the maximizer of p on $\mathbb{R}$, if it exists. Consider the beta density

$$p(x) = \begin{cases} cx^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant and $\alpha, \beta > 1$. This density has a unique mode. Obtain it as a function of the parameters. Explain your derivation. In particular, justify your claim that the point you obtain is the mode.

Solution 1. If a point x is a maximizer of p then it must be that $x \in (0,1)$, since p is strictly positive on this interval. Since p is differentiable on $(0,1)$, any maximizer must be a stationary point in this inverval. Hence the set of maximizers is a subset of the set of stationary points of p in $(0,1)$. Differentiating p and setting the result equal to zero gives the equation

$$x = \gamma(1-x) \quad \text{where} \quad \gamma := \frac{\alpha-1}{\beta-1}.$$

This equation has the unique solution

$$x^* = \frac{\gamma}{1+\gamma} = \frac{\alpha-1}{\alpha+\beta-2}.$$

Since this point is the only stationary point, it is the only candidate for a maximizer. In the question we are told that a maximizer exists, so x^* must be the maximizer. In other words, x^* is the mode.

Question 2. Let $\mathbf{D}$ be the 10×10 diagonal matrix $\text{diag}(1, 2, \ldots, 10)$.

(i) What is $\mathbf{D}^2$?

(ii) Is $\mathbf{D}$ invertible? If so, what is the inverse?

Solution 2. We know from the lecture slides that the k-th power of $\mathbf{D}$ is the diagonal matrix formed by taking the k-th power of the element along the principle diagonal of $\mathbf{D}$. That is,

$$\mathbf{D}^2 = \text{diag}(1^2, 2^2, \dots, 10^2) = \text{diag}(1, 4, \dots, 100).$$

We also know that $\mathbf{D}$ is invertible (since all elements on the principle diagonal are nonzero), and that the inverse is

$$\mathbf{D}^{-1} = \text{diag}(1/1, 1/2 \dots, 1/10).$$

Question 3. What is the dimension of $\mathbb{R}^N$? Explain your answer, using the definition of dimension.

Solution 3. The dimension of $\mathbb{R}^N$ is N. The explanation is as follows: The dimension of $\mathbb{R}^N$ is the number of elements in any basis of $\mathbb{R}^N$. A set of vectors is a basis for a linear subspace if it is linearly independent and spans that subspace. Hence it suffices to find N linearly independent vectors that span $\mathbb{R}^N$. The canonical basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_N$ are such a set.

Question 4. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same shape. Show that if $\mathbf{A}$ is singular then so is $\mathbf{C} := \mathbf{AB}$.

Solution 4. By the rules for determinants, $\det(\mathbf{C}) = \det(\mathbf{A}) \det(\mathbf{B})$. Since $\mathbf{A}$ is singular, the right hand size and therefore the left hand side is zero. Since a zero determinant implies singularity, we conclude that $\mathbf{C}$ is singular.

Question 5. Consider the matrix

$$\mathbf{B} := \begin{pmatrix} 1 & 4 & 0 \\ 0 & 8 & 1 \end{pmatrix}$$

(ij Find a basis for the column space (i.e., the span of the columns) of $\mathbf{B}$. Explain your derivation.

(iij What is the rank of $\mathbf{B}$? Explain your answer.

Solution 5. Regarding part (i), the column space of $\mathbf{B}$ is denoted span($\mathbf{B}$) and defined as the span of its columns. Since the columns are vectors in $\mathbb{R}^2$, the span of the columns is a subset of $\mathbb{R}^2$. The first and last columns together are the canonical basis vectors for $\mathbb{R}^2$. We know that these two vectors span $\mathbb{R}^2$. The span of a larger set is at least as large. Hence the span of all three vectors is all of $\mathbb{R}^2$. The first two vectors of $\mathbf{B}$ form a basis for $\mathbb{R}^2$, and hence of span($\mathbf{B}$), since they are linearly independent and span $\mathbb{R}^2$.

Regarding part (ii), the rank of $\mathbf{B}$ is 2, since the dimension of the column space is the number of elements in this (or any other) basis, which is 2.

Question 6. Let $\mathbf{A}$ be any $N \times K$ matrix, let λ be a real number, and let $\mathbf{B} := \mathbf{A}'\mathbf{A} + \lambda \mathbf{I}$ where $\mathbf{I}$ is the $K \times K$ identity.

(i) Show that $\mathbf{B}$ is symmetric.

(ii) Show that $\mathbf{B}$ is positive definite whenever $\lambda > 0$.

Solution 6. $\mathbf{B}$ is symmetric, because $\mathbf{A}'\mathbf{A}$ and $\mathbf{I}$ are symmetric. In particular,

$$\mathbf{B}' = (\mathbf{A}'\mathbf{A} + \lambda \mathbf{I})' = (\mathbf{A}'\mathbf{A})' + \lambda \mathbf{I}' = \mathbf{A}'\mathbf{A} + \lambda \mathbf{I} = \mathbf{B}$$

To show that $\mathbf{B}$ is positive definite when $\lambda > 0$, we need to show that if $\mathbf{x} \in \mathbb{R}^K$ and $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}'\mathbf{B}\mathbf{x} > 0$. To see that this is the case, take such an $\mathbf{x}$ and observe that

$$\begin{aligned}
\mathbf{x}'\mathbf{B}\mathbf{x} &= \mathbf{x}'(\mathbf{A}'\mathbf{A} + \lambda \mathbf{I})\mathbf{x} \\
&= \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} + \lambda \mathbf{x}'\mathbf{I}\mathbf{x} \\
&= \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} + \lambda \|\mathbf{x}\|^2 \\
&= (\mathbf{A}\mathbf{x})'\mathbf{A}\mathbf{x} + \lambda \|\mathbf{x}\|^2 \\
&= \|\mathbf{A}\mathbf{x}\|^2 + \lambda \|\mathbf{x}\|^2
\end{aligned}$$

The first term on the right-hand side is nonnegative, and the second term is strictly positive, because $\lambda > 0$ and $\mathbf{x} \neq \mathbf{0}$. Hence $\mathbf{B}$ is positive definite, as claimed.

Question 7. Let $\mathbf{A}$ be a square matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_N$. Show that if $\mathbf{a}_i'\mathbf{a}_j = \mathbb{1}\{i = j\}$, then $\mathbf{A}'$ is the inverse of $\mathbf{A}$.

Solution 7. To show that $\mathbf{A}'$ is the inverse of $\mathbf{A}$, we need to show that $\mathbf{A}'\mathbf{A} = \mathbf{I}_N$. By the definition of matrix multiplication, $\mathbf{A}'\mathbf{A}$ is the matrix such that the i, j-th element is $\mathbf{a}_i'\mathbf{a}_j$. By the assumption $\mathbf{a}_i'\mathbf{a}_j = \mathbb{1}\{i = j\}$, this matrix is equal to $\mathbf{I}_N$.

Question 8. Let $\mathbf{A}$ be positive definite. Show that $\text{trace}(\mathbf{A}) > 0$.

Solution 8. The trace of $\mathbf{A}$ is the sum of the diagonal elements of $\mathbf{A}$, and will be strictly positive if all of the diagonal elements are strictly positive. This must be the case, because if $\mathbf{e}_n$ is the n-th canonical basis vector, then $\mathbf{e}_n \neq \mathbf{0}$ and hence $\mathbf{e}_n'\mathbf{A}\mathbf{e}_n > 0$. But $\mathbf{e}_n'\mathbf{A}\mathbf{e}_n = a_{nn}$. Hence $a_{nn} > 0$ for all n.

Question 9. Let Ω be a sample space, let $\mathbb{P}$ be a probability on Ω, and let A and B be events. Show that $\mathbb{P}(A) = \mathbb{P}(B) = 1$ implies $\mathbb{P}(A \cap B) = 1$.

Solution 9. Let $\mathbb{P}$, A and B be as stated in the question, with $\mathbb{P}(A) = \mathbb{P}(B) = 1$. From the lecture slides we know that

$$\mathbb{P}(A^c \cup B^c) \leq \mathbb{P}(A^c) + \mathbb{P}(B^c) = 0.$$

$$\therefore \quad \mathbb{P}((A \cap B)^c) = 0.$$

$$\therefore \quad \mathbb{P}(A \cap B) = 1.$$

Question 10. Let Ω be any sample space, and let $\mathcal{F}$ be the set of events. Define $\mathbb{P} \colon \mathcal{F} \to [0, 1]$ by $\mathbb{P}(A) = 1$ if A is nonempty and $\mathbb{P}(\emptyset) = 0$. Is $\mathbb{P}$ a probability on $\mathcal{F}$? Why or why not?

Solution 10. In general, $\mathbb{P}$ is not a probability, because if A and B are disjoint nonempty sets, then $A \cup B$ is nonempty, and hence $\mathbb{P}(A \cup B) = 1$ while $\mathbb{P}(A) + \mathbb{P}(B) = 1 + 1 = 2$. Therefore additivity does not hold.

(The only caveat to this argument is that we may not be able to select two disjoint nonempty sets. This occurs precisely when Ω has only one element. If Ω has only one element, then additivity cannot be contradicted, and $\mathbb{P}$ is a probability. Students are not expected to notice this and it is not part of the marks.)

Question 11. Let X be a random variable on sample space Ω with

$$\text{rng}(X) = \{x_1, \ldots, x_K\}$$

where $x_1 < x_2 < \ldots < x_K$. Show that the events

$$\{X = x_k\}, \quad k = 1, \ldots, K$$

form a partition of Ω.

Solution 11. Let

$$E_k := \{X = x_k\} := \{\omega \in \Omega : X(\omega) = x_k\}$$

To show that $E_1, \ldots, E_K$ is a partition, we need to show that the sets are disjoint and that their union is Ω. To see that the sets are disjoint, observe that if $k \neq j$ and $\omega \in E_k$, then $X(\omega) = x_k$ and hence $X(\omega) \neq x_j$. It follows that $\omega \notin E_j$. A similar argument shows that if $\omega \in E_j$ then $\omega \notin E_k$. Hence the sets are disjoint.

In addition, since $\{x_1, \ldots, x_K\}$ is the range of X, and since X is a function on Ω, it must be that, given any $\omega \in \Omega$, we have $X(\omega) = x_k$ for some k in $1, \ldots, K$. Hence $\omega \in E_k$. In other words, every $\omega \in \Omega$ must be in one of the sets $E_1, \ldots, E_K$. Hence their union is all of Ω.

Question 12. Let X be a binary random variable with $\mathbb{P}\{X = 1\} = p$.

(i) Calculate the expectation of X.

(ii) Calculate the variance of X.

Solution 12. The expectation of X is $0 \times \mathbb{P}\{X = 0\} + 1 \times \mathbb{P}\{X = 1\} = p$. The variance is then

$$\begin{aligned}
\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right] &= \mathbb{E}\left[(X - p)^2\right] \\
&= (0 - p)^2 \times \mathbb{P}\{X = 0\} + (1 - p)^2 \times \mathbb{P}\{X = 1\} \\
&= p^2(1 - p) + (1 - p)^2 p \\
&= p(1 - p)
\end{aligned}$$

Question 13. Let $\theta > 0$ and let X be a random variable uniformly distributed (i.e., having the uniform distribution) on the interval $(0, \theta)$.

(i) Give an expression for the density of X.

(ii) Calculate the expectation of X.

(iii) Calculate the variance of X.

Show your derivation for parts (ii) and (iii).

Solution 13. The density of X is the uniform density

$$p(x) = \frac{1}{\theta}\mathbb{1}\{0 < x < \theta\}$$

The expectation of X is therefore

$$\int_{-\infty}^{\infty} x p(x) dx = \int_0^{\theta} x \frac{1}{\theta} dx = \frac{1}{\theta}\frac{\theta^2}{2} = \frac{\theta}{2}$$

The variance of X is

$$\int_{-\infty}^{\infty} \left(x - \frac{\theta}{2}\right)^2 p(x) dx = \int_0^{\theta} \left(x - \frac{\theta}{2}\right)^2 \frac{1}{\theta} dx = \frac{\theta^2}{12}$$

Question 14. Let $\mathbf{X} \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $N(\mathbf{0}, \mathbf{\Sigma})$ is the multivariate normal distribution in $\mathbb{R}^2$ with mean equal to the origin $\mathbf{0} \in \mathbb{R}^2$ and variance-covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Let $\mathbf{Y} := \mathbf{a} + \mathbf{B}\mathbf{X}$ where

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix}$$

(i) What is the distribution of $\mathbf{Y}$? Give a full description of the distribution with explanation.

(ii) Are Y_1 and Y_2 independent? Why or why not?

Solution 14. Since linear combinations of multivariate normals are multivariate normal, $\mathbf{Y}$ is multivariate normal. To fully describe its distribution we need to pin down the mean and variance covariance matrix. By the rules for multivariate expectations and variances, we have

$$\mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{a} + \mathbf{B}\mathbf{X}] = \mathbf{a} + \mathbf{B}\mathbb{E}[\mathbf{X}] = \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For the variance-covariance matrix we have the rule

$$\operatorname{var}[\mathbf{a} + \mathbf{B}\mathbf{X}] = \mathbf{B}\operatorname{var}[\mathbf{X}]\mathbf{B}'$$

which in this case gives

$$\begin{aligned} \operatorname{var}[\mathbf{Y}] &= \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix} \\ &= 2 \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 32 & 32 \\ 32 & 34 \end{pmatrix} \end{aligned}$$

Regarding part (ii), Y_1 and Y_2 are not independent, since their covariance is $32 \neq 0$.

Question 15. Let X be any random variable with $\mathbb{E}[|X|] < \infty$. Show that $X_n := X/n$ converges to zero in probability as $n \to \infty$.

Solution 15. To show that $X_n \xrightarrow{p} 0$ we need to show that for any $\delta > 0$ we have

$$\mathbb{P}\{|X_n - 0| > \delta\} = \mathbb{P}\{|X/n| > \delta\} \to 0 \qquad (n \to \infty)$$

To show this we can use an inequality from the lecture slides (the Markov inequality) to get

$$\mathbb{P}\{|X/n| > \delta\} = \mathbb{P}\{|X| > n\delta\} \leq \frac{\mathbb{E}|X|}{n\delta} \to 0$$

Question 16. Consider the system of equations $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{W}$ where

- $\mathbf{B}$ is $N \times K$ and $N > K$
- $\mathbf{A}$ is $N \times N$
- $\mathbf{X}$ and $\mathbf{Y}$ are $N \times 1$
- $\mathbf{W}$ is $K \times 1$

Suppose we are able to observe both $\mathbf{Y}$ and $\mathbf{X}$, and we know the matrices $\mathbf{A}$ and $\mathbf{B}$. On the other hand, $\mathbf{W}$ is a vector of unobservable shocks. Our aim is to solve for $\mathbf{W}$ in terms of the observable matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$ and $\mathbf{Y}$. Show that if $\mathbf{B}$ has rank K then this is possible and give the expression for $\mathbf{W}$ in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$ and $\mathbf{Y}$.

Solution 16. If we premultiply $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{W}$ by $\mathbf{B}'$ we get

$$\mathbf{B}'\mathbf{Y} = \mathbf{B}'\mathbf{A}\mathbf{X} + \mathbf{B}'\mathbf{B}\mathbf{W},$$

or

$$\mathbf{B}'\mathbf{B}\mathbf{W} = \mathbf{B}'(\mathbf{Y} - \mathbf{A}\mathbf{X}).$$

We can now write

$$\mathbf{W} = (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'(\mathbf{Y} - \mathbf{A}\mathbf{X}).$$

provided that $\mathbf{B}'\mathbf{B}$ is invertible. To show this it suffices to show the $\mathbf{B}'\mathbf{B}$ is positive definite. To show that this is true, take any $\mathbf{x} \neq \mathbf{0}$. Note that, since $\mathbf{B}$ has full column rank its columns are linearly independent, and, since $\mathbf{x} \neq \mathbf{0}$, we must have $\mathbf{B}\mathbf{x} \neq \mathbf{0}$. Hence

$$\mathbf{x}'\mathbf{B}'\mathbf{B}\mathbf{x} = \mathbf{x}'\mathbf{B}'\mathbf{B}\mathbf{x} = (\mathbf{B}\mathbf{x})'(\mathbf{B}\mathbf{x}) = \|\mathbf{B}\mathbf{x}\|^2 > 0.$$

This proves positive definiteness and hence invertibility of $\mathbf{B}'\mathbf{B}$.