# Structural Estimation of Directional Dynamic Games With Multiple Equilibria

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Econometric Society World Congress
20 August 2025, Seoul

# Directionality $\Longrightarrow$ Full solution $\Longrightarrow$ Nested MLE

- Focus on the subclass of discrete stochastic games with directional state transitions $\rightarrow$ directional dynamic games
- State recursion algorithm = generalization of backward induction to solve for MPE in stages
- Multiple solutions at stage games $\rightarrow$ RLS algorithm to enumerate all combinations of solutions $\iff$ solve for all MPEs (IRS, 2016)
- This paper develops a nested full solution MLE estimator $\equiv$ NRLS estimator based on integer programming branch-n-bound algorithm
  - Fully robust to multiplicity of equilibria in the model and the data
  - Computationally feasible
  - Computational burden decreases with sample size
- Extensive Monte Carlo study of existing estimators: 2-step, NPL, EPL, MPEC vs NRLS

Iskhakov, Rust and Schjerning (2016, ReStud)
Recursive Lexicographical Search: Finding All Markov Perfect Equilibria of Finite State Directional Dynamic Games

# MLE for dynamic games with multiple equilibria

- Data from $M$ independent markets from $T$ periods, $N$ players

$$Z = \left\{ a^{ijt}, x^{it} \right\}_{i \in \{1 \ldots M\}, j \in \{1, \ldots, N\}, t \in \{1, \ldots, T\}}$$

- MPE is a pair of strategy profiles and value functions such that

$$\begin{aligned} V_\theta &= \Psi^{V_\theta}(V_\theta, P_\theta, \theta) && \text{(Bellman equations)} \\ P_\theta &= \Psi^{P_\theta}(V_\theta, P_\theta, \theta) && \text{(CCPs = mutual best responces)} \end{aligned}$$

- Multiplicity $\rightarrow$ set of equilibria $\mathcal{E}(\theta) = \{V_\theta^k, P_\theta^k\}_{k \in \{1, \ldots, K(\theta)\}}$
- MLE estimator $\hat{\theta}^{ML}$ is given by

$$\hat{\theta}^{ML} = \arg\max_\theta \left[ \max_{k \in \{1, \ldots, K(\theta)\}} \sum_{t=1}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} \log P_j^k(a^{ijt} | x^{it}; \theta) \right]$$

- One equilibrium in the data, relax later with grouped fixed effects
- Inner loop requires full solution, impossible?

# MLE by contrained optimization (MPEC)

- Idea: use discretized values of $P$ and $V$ as *variables*
- Augmented log-likelihood function

$$\mathcal{L}(Z, P, \theta) = \sum_{t=1}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} \log P_j(a^{ijt} | x^{it}; \theta)$$

- The constrained optimization formulation of the ML estimation problem is

$$\max_{\theta, P, V} \mathcal{L}(Z, P, \theta) \text{ subject to } \begin{cases} V = \Psi^V(V, P, \theta) \\ P = \Psi^P(V, P, \theta) \end{cases}$$

- May work with multiple equilibria with smart optimization algorithms
- Much bigger computational problem
- Implements the same MLE estimator (*when it works*)

Su (2013); Egesdal, Lai and Su (2015)

# Other existing estimation methods

- Two step (CCP) estimators
  - Fast, do not impose equilibrium constraints, finite sample bias
  1. Estimate CCP $\rightarrow \hat{P}$
  2. Method of moments • Minimal distance • Pseudo likelihood

  Hotz, Miller (1993); Altug, Miller (1998); Pakes, Ostrovsky, and Berry (2007); Pesendorfer, Schmidt-Dengler (2008)

- Nested pseudo-likelihood (NPL)
  - Recursive two step pseudo-likelihood
  - Bridges the gap between efficiency and tractability
  - Unstable under multiplicity

  Aguirregabiria, Mira (2007); Aguirregabiria, Marcoux (2021)

- Efficient pseudo-likelihood (EPL)
  - Incorporates Newton step in the NPL operator
  - More robust to the stability and multiplicity of equilibria

  Dearing, Blevins (2024)

# Directional dynamic games (DDG)

DDG is a finite state stochastic game where state transitions under all feasible Markovian strategies form a directional acyclic graph with self-loops
(see IRS, 2016 for formal definition)

- points of finite state space $X \rightarrow$ vertexes of the graph
- Edge from $x_i$ to $x_j \iff$ under some strategy profile the hitting probability of $x_j$ from $x_i$ is positive

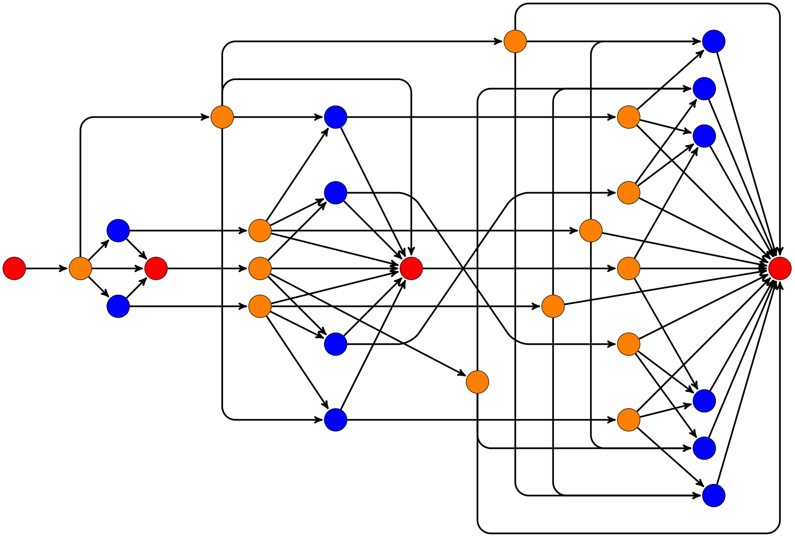
1. Simple algorithm to determine if graph is a DAG
2. Topological sort to find totally ordered partition
3. Backward recursion on the found total order

DDGs in the literature:

Judd, Schmedders, Yeltekin (2012); Dube, Hitsch, Chintagunta (2010); Iskhakov, Rust, Schjerning (2018); Anderson, Rosen, Rust, Wong (2024)

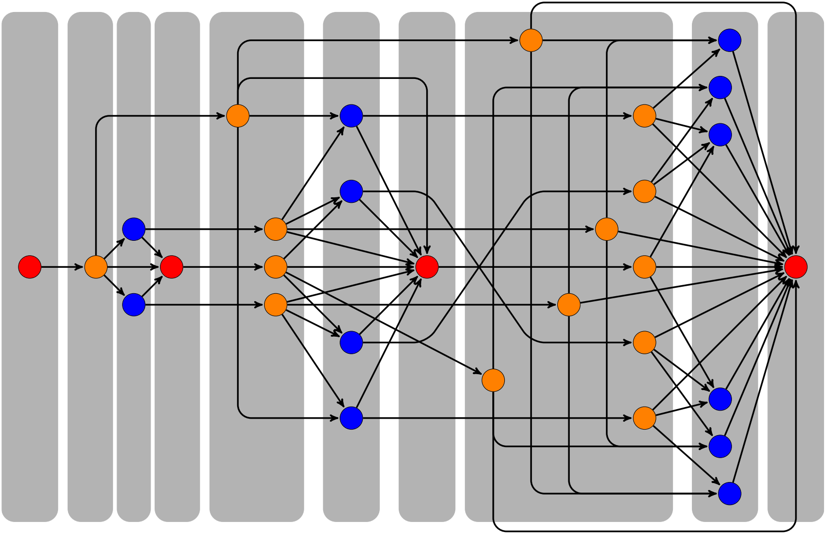
# Graph representation of all possible transitions on $X$

## Transitions induced by all feasible strategy profiles



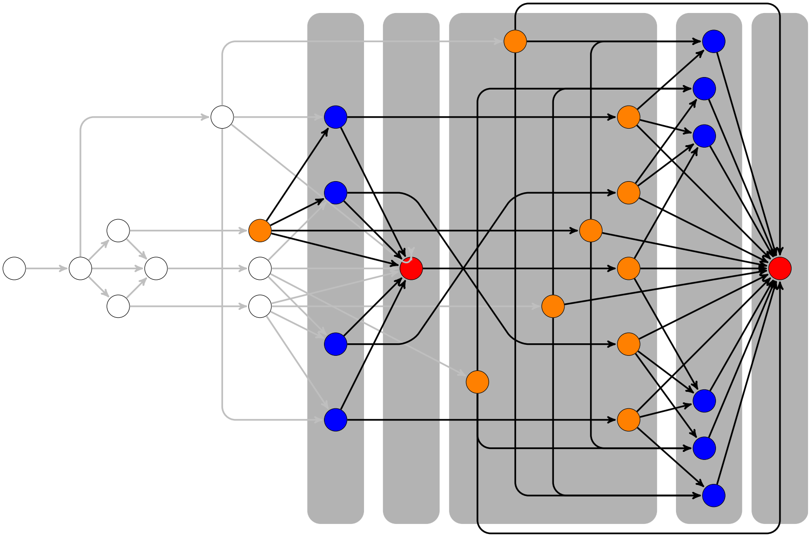
# Total order on the partition of the state space

## After running a topoligical sort algorithm on the DAG



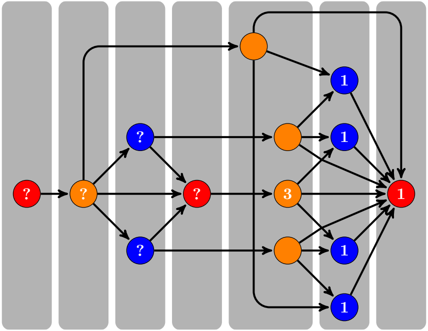
# State recursion = backward induction on the state space

## Solving subgames in continuation strategies → many small problems

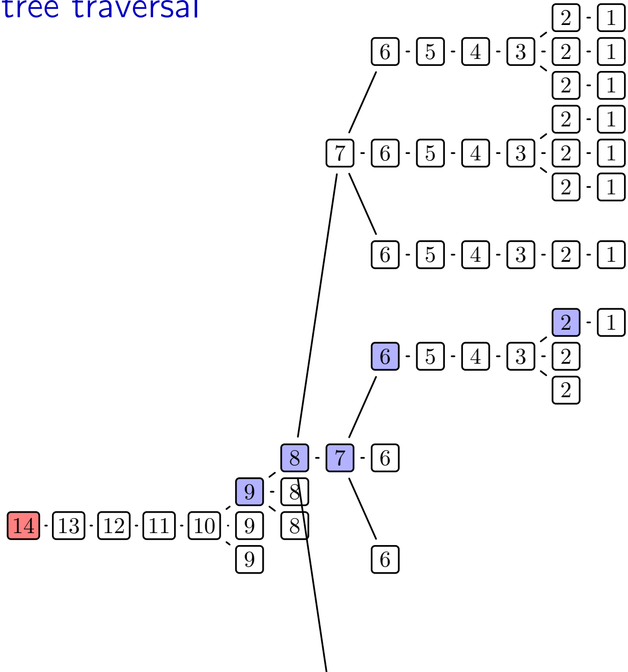


# Multiplicity of stage equibiria $\Longleftrightarrow$ Multiplicity of MPE

- State recursion proceeds conditional on equilibrium selection rule
- Selected equilibrium at downstream stage affects the equilibria and number of equilibria at upstream stages
- Need to systematically combine different stage equilibria $\rightarrow$ Recursive Lexicographical Search $\equiv$ Depth-first tree traversal



# RLS = tree traversal



# Nested Recursive Lexicographical Search (NRLS)

- Data from $M$ independent markets from $T$ periods, $N$ players

$$Z = \left\{a^{ijt}, x^{it}\right\}_{i\in\{1\ldots M\}, j\in\{1,\ldots,N\}, t\in\{1,\ldots,T\}}$$

- Set of equilibria $\mathcal{E}(\theta) = \{V_\theta^k, P_\theta^k\}_{k\in\{1,\ldots,K(\theta)\}}$

1. Outer loop Maximization of the likelihood function w.r.t. to structural parameter $\theta$

$$\theta^{ML} = \arg\max_{\theta\in\Theta} \mathcal{L}(Z,\theta)$$

2. Inner loop Maximization of the likelihood function w.r.t. equilibrium selection $\equiv$ discrete parameter $k \in \{1,\ldots,K(\theta)\}$

$$\mathcal{L}(Z,\theta) = \arg\max_{k\in\{1,\ldots,K(\theta)\}} \mathcal{L}(Z,\theta,P_\theta^k)$$

- With multiple equilibria in the data $\mathcal{L}(Z,\theta)$ has more elaborate form

# Likelihood over the state space

## Can efficiently represent likelihood by counts of observations

- With equilibrium $k$ choice probabilities $P_j^k(a|x;\theta)$, likelihood is

$$\mathcal{L}(Z,\theta,P_\theta^k) = \sum_{t=1}^{T}\sum_{i=1}^{M}\sum_{j=1}^{N} \log P_j^k(a^{ijt}|x^{it};\theta)$$

- Let $\iota$ index points in the state space
  $\iota = 1$ in the initial subset, $\iota = |X|$ in the terminal subset of $X$
- Denote $n_\iota$ the number of observations in state $x_\iota$ and $n_\iota^{a_j}$ the number of observations of player $i$ taking action $a_j$ at $x_\iota$

$$n_\iota = \sum_{t=1}^{T}\sum_{i=1}^{M} \mathbb{1}\{x^{it} = x_\iota\} \qquad n_\iota^{a_j} = \sum_{t=1}^{T}\sum_{i=1}^{M} \mathbb{1}\{a^{ijt} = a_j, x^{it} = x_\iota\}$$

- Then equilibrium-specific likelihood is given by

$$\mathcal{L}(Z,\theta,P_\theta^k) = \sum_{\iota=1}^{|X|}\sum_{j=1}^{N}\sum_{a_j} n_\iota^{a_j} \log P_j^k(a_j|x_\iota;\theta)$$

# Branch and bound solution method

Land and Doig, 1960 *Econometrica*

- Old method for solving integer programming problems

- **Branching**: RLS tree
- **Bounding**: The bound function is partial likelihood of equilibrium $k$ calculated on the subset of states $\iota \in \mathcal{S} \subset X$

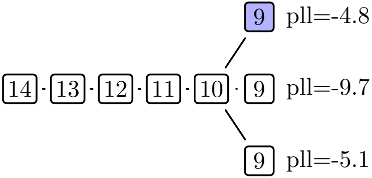
$$\mathcal{L}^{\text{part}}(Z^{\mathcal{S}}, \theta, P_\theta^k) = \sum_{\iota \in \mathcal{S}} \sum_{j=1}^{N} \sum_{a_j} n_\iota^{a_j} \log P_j^k(a_j | x_\iota; \theta)$$

- Where $Z^{\mathcal{S}} = \{(a, x) : x \in \mathcal{S}\}$ denotes data observed on $\mathcal{S}$
- Monotonic decreasing in cardinality of $\mathcal{S}$ (declines as more data is added)
- Equals to the full log-likelihood on the full state space when $Z^{\mathcal{S}} = Z$ (at the leafs of RLS tree, next slide)

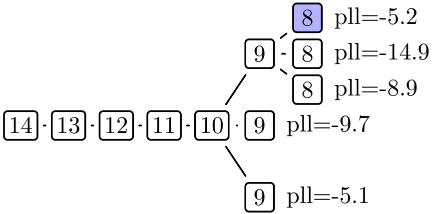
# BnB on RLS tree, step 1

14 · 13 · 12 · 11 · 10 Partial loglikelihood = -3.2

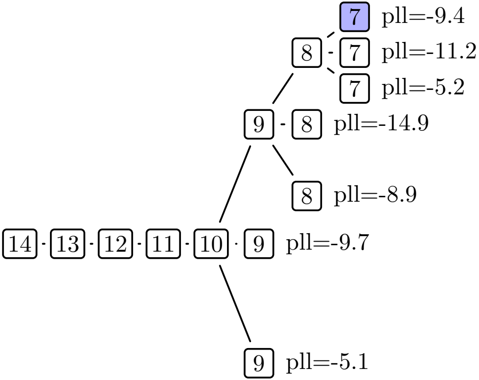
# BnB on RLS tree, step 2



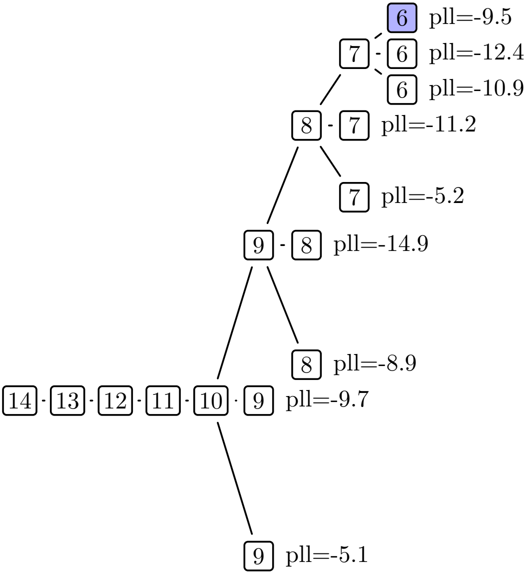
# BnB on RLS tree, step 3



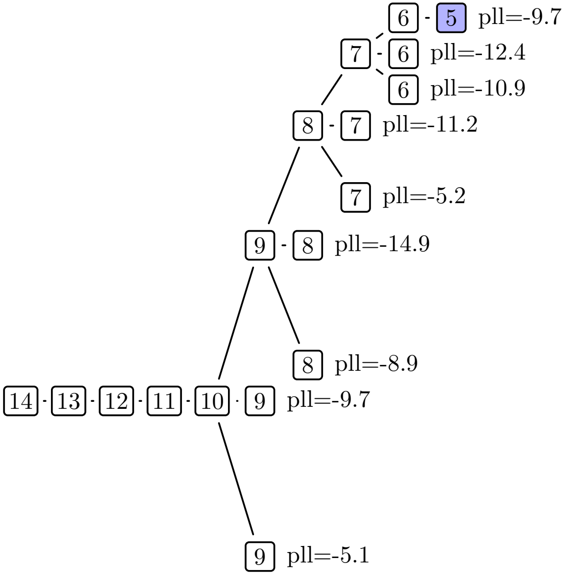
# BnB on RLS tree, step 4



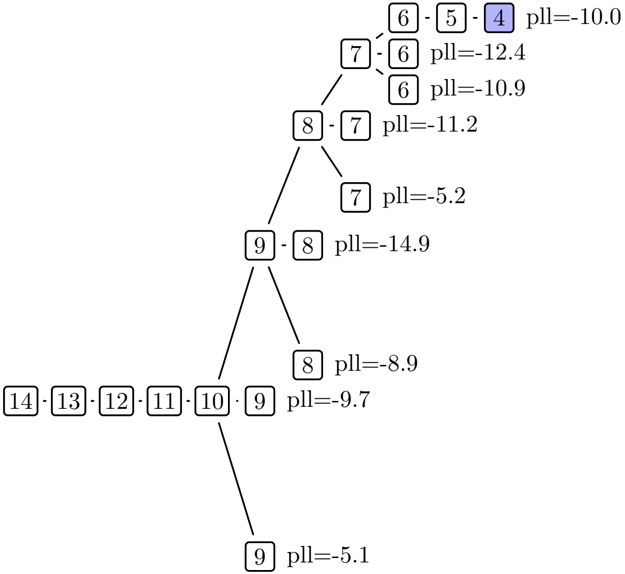
# BnB on RLS tree, step 5



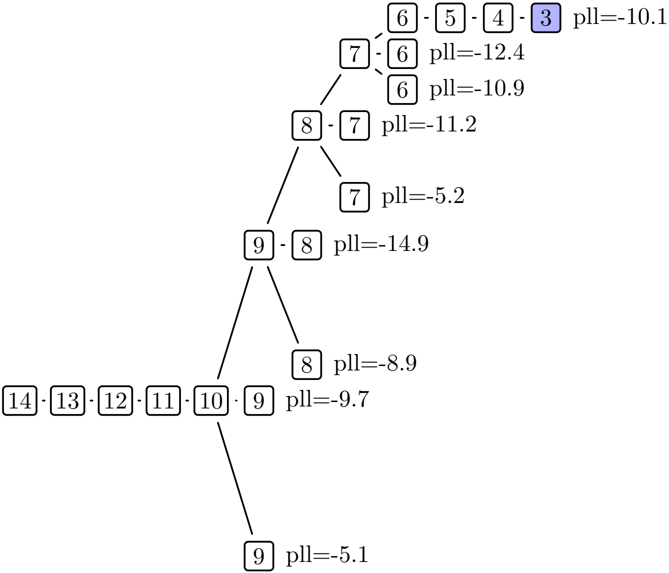
# BnB on RLS tree, step 6



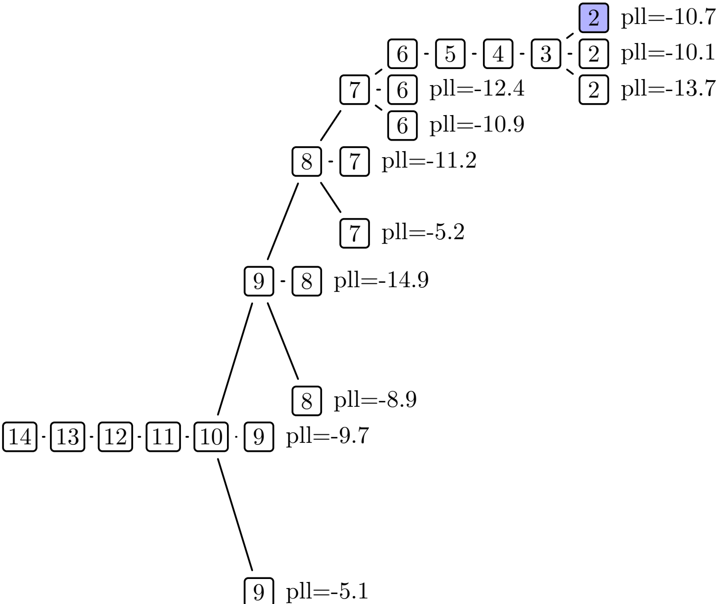
# BnB on RLS tree, step 7



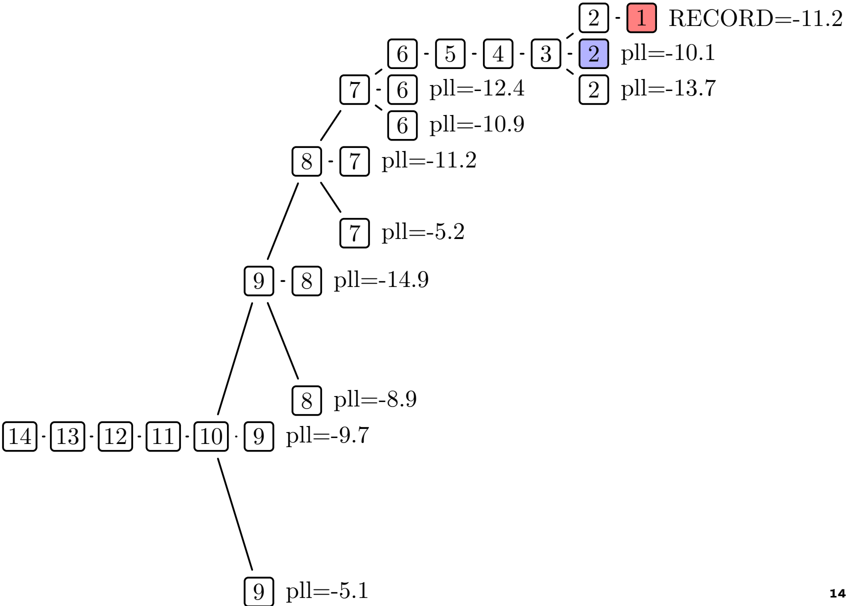
# BnB on RLS tree, step 8



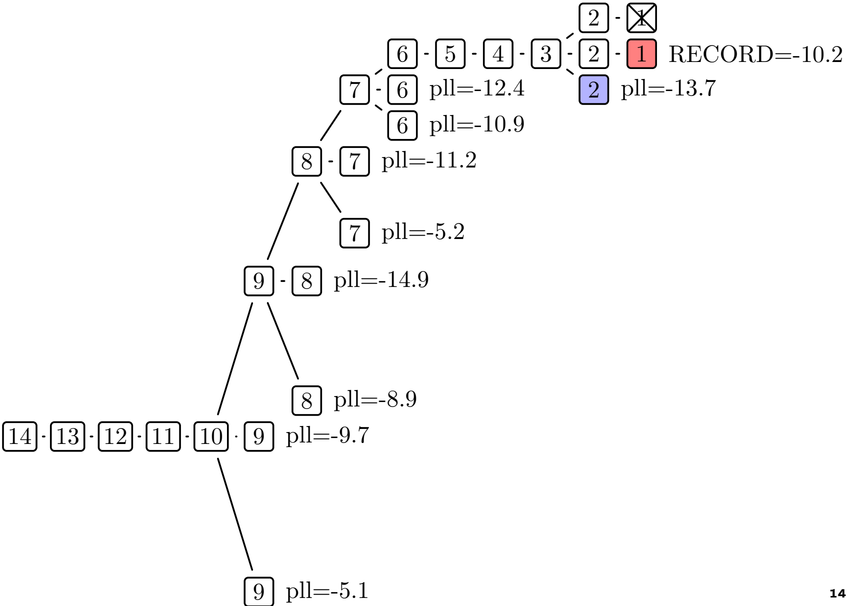
# BnB on RLS tree, step 9



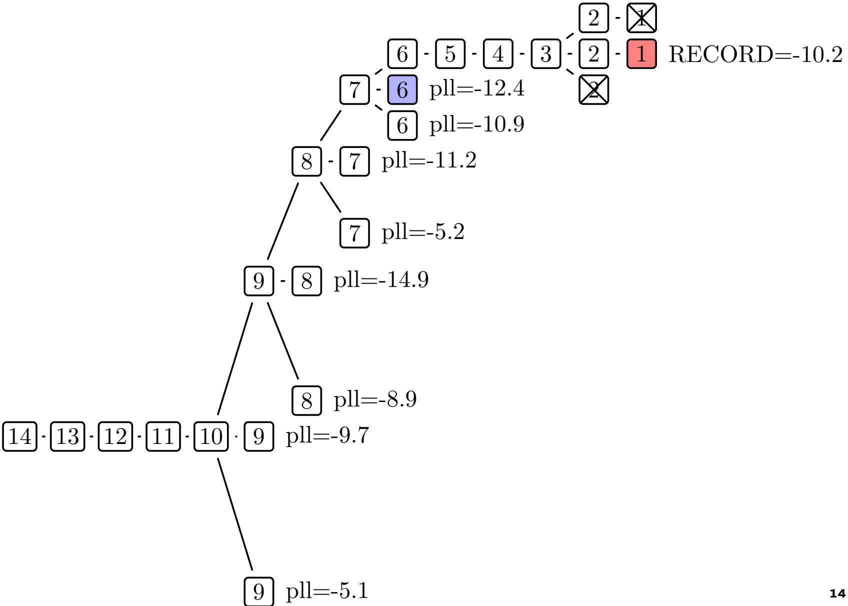
# BnB on RLS tree, step 10



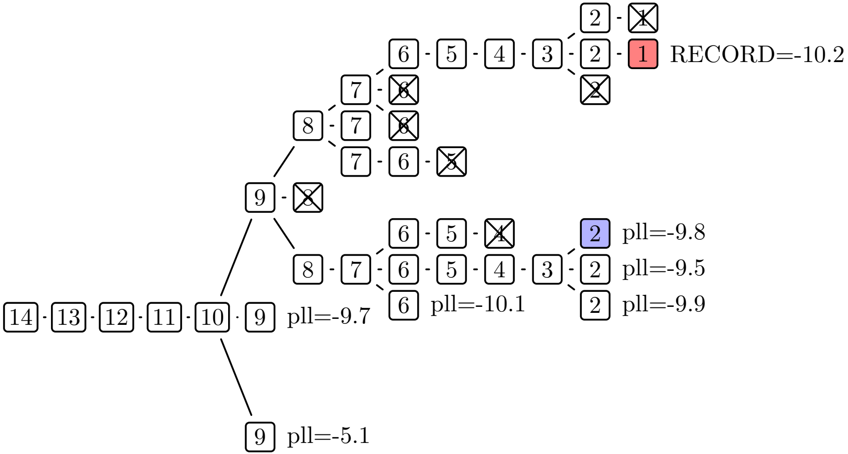
# BnB on RLS tree, step 11



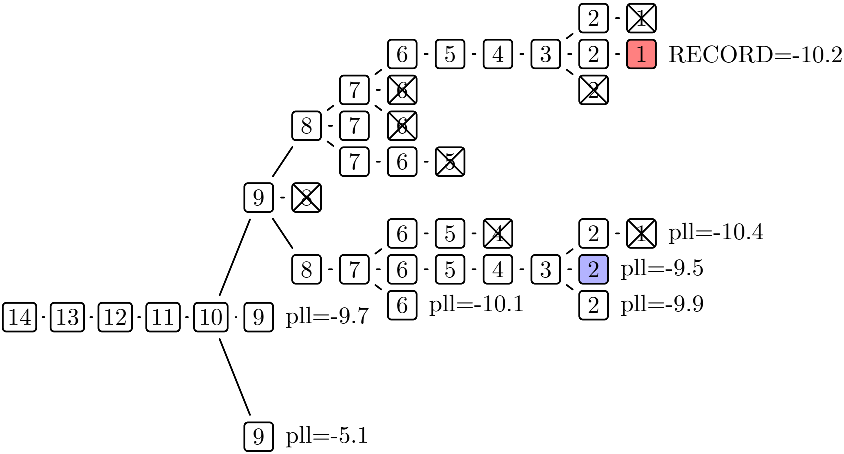
# BnB on RLS tree, step 12



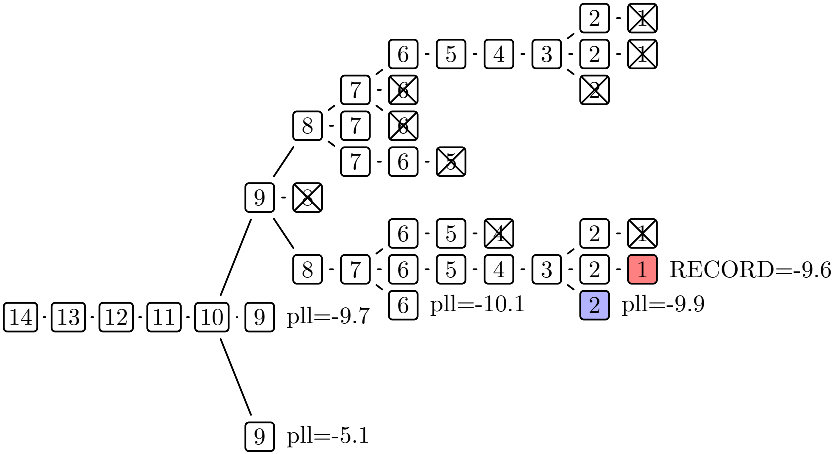
# BnB on RLS tree, step 28



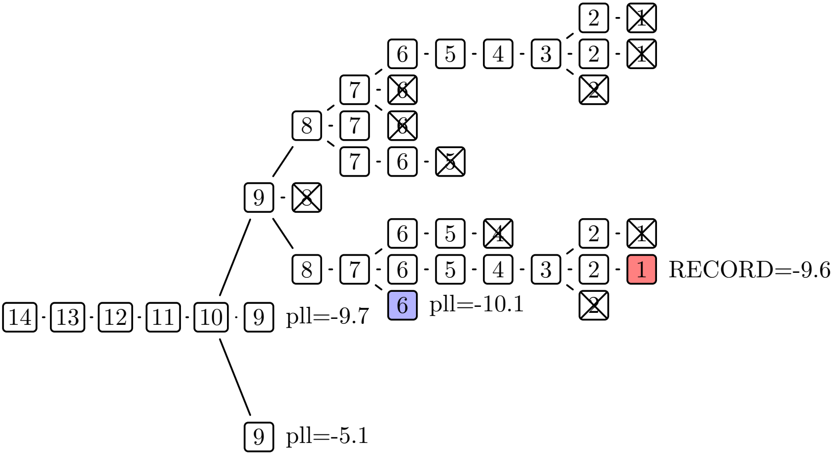
# BnB on RLS tree, step 29



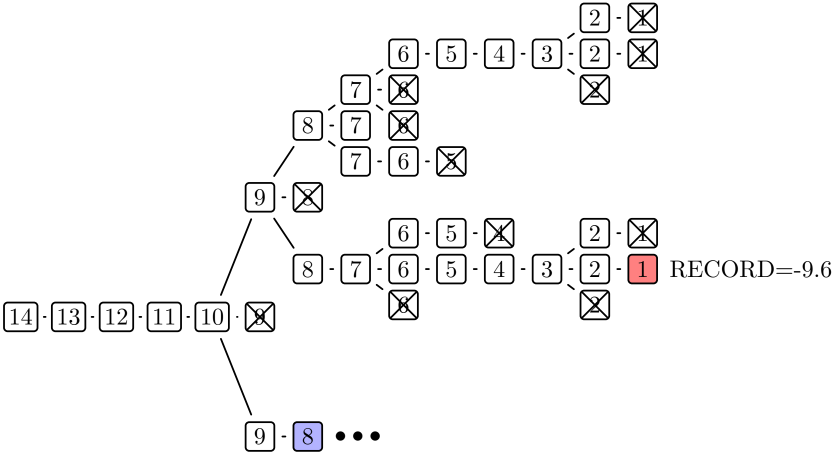
# BnB on RLS tree, step 30



# BnB on RLS tree, step 31



# BnB on RLS tree, step 34



# Refinement: non-parametric likelihood bounding

Much more powerful bound for BnB based on empirical frequencies

- Replace choice probabilities $P_j^k(a_j|x_\iota;\theta)$ with frequencies $n_\iota^{a_j}/n_\iota$

$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) = \sum_{\iota\in\mathcal{S}} \sum_{i=1}^{J} \sum_{a} n_\iota^{a_i} \log(n_\iota^{a}/n_\iota)$$

- $\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}})$ depends only on the counts from the data!
- Not hard to show algebraically that for any $Z^{\mathcal{S}}$ (≈Gibbs inequality)

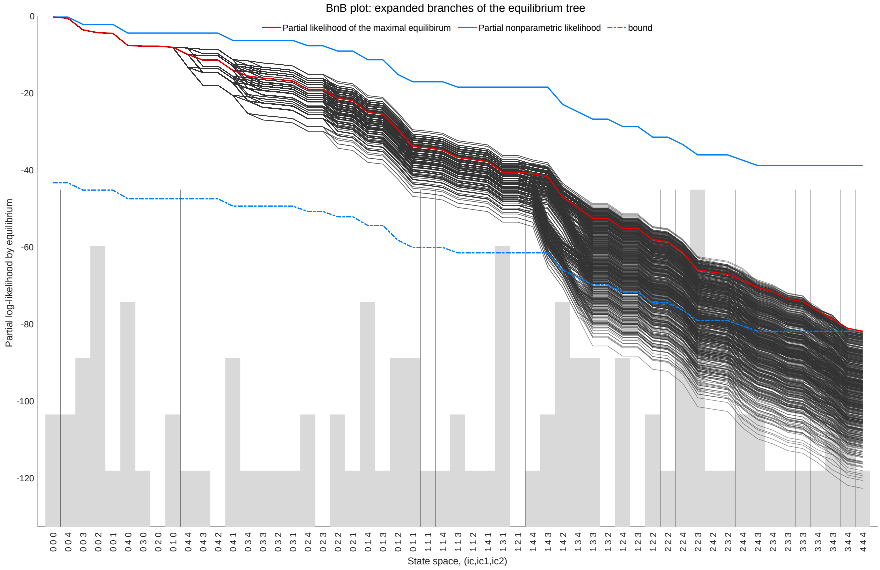
$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) > \mathcal{L}^{\text{part}}(Z^{\mathcal{S}}, \theta, P^k)\ \forall\theta, k$$

- Therefore partial likelihood can be optimistically extrapolated by empirical likelihood at any step $\iota$ of the RLS tree traversal

$$\mathcal{L}^{\text{part}}(Z^{\{|X|,\dots,\iota\}}, \theta, P^k) + \mathcal{L}^{\text{non-par}}(Z^{\{\iota-1,\dots,1\}})$$

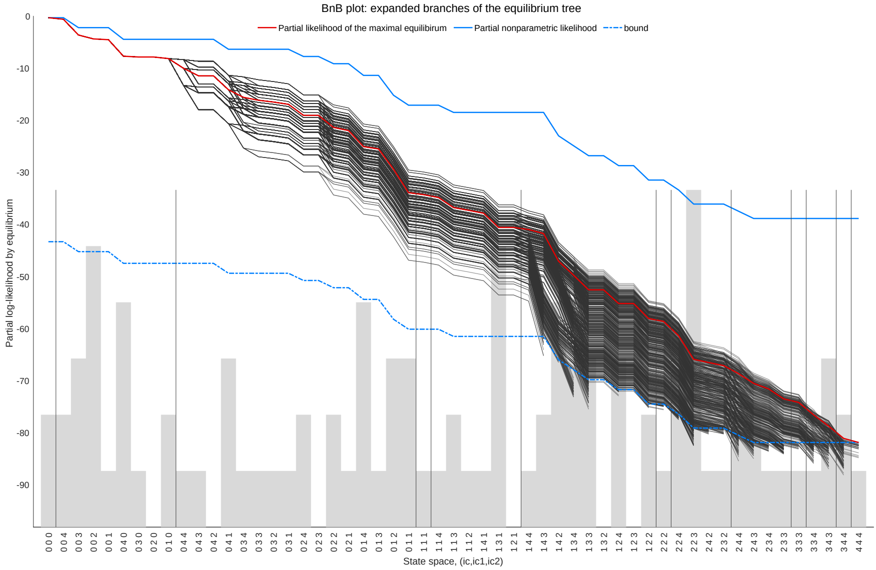
# Non-parameteric likelihood bounding

$\iota = |X| = 14$ (terminal state) on the left, $\iota = 1$ (initial state) on the right



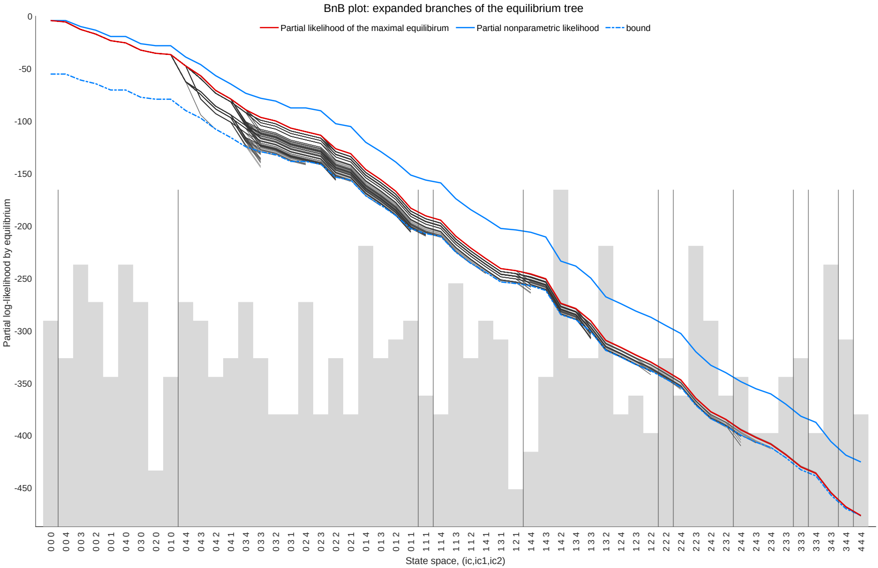
# BnB with non-parameteric likelihood bound

## Greedy traversal + non-parameteric likelihood bound



# BnB with non-parameteric likelihood bound, larger sample

## Non-parametric → parametric likelihood as $N \to \infty$ at true $\theta \Rightarrow$ even less computation



# BnB refinement with non-parametric likelihood

- For any amount of data the non-parametric likelihood is greater or equal to the parametric likelihood *algebraically*
- BnB augmented with non-parameteric likelihood bound gives sharper Bounding Rules $\rightarrow$ less computation

- Wih more data as $M \rightarrow \infty$, $T \rightarrow \infty$
- Non-parametric log-likelihood converges to the actual likelihood
- The width of the band between the blue lines in the plots decreases

  $\rightarrow$ Even sharper Bounding Rules
  $\rightarrow$ Even less computation

BnB yields exact solution of the inner integer maximization problem
$\Longrightarrow$ MLE for any sample size, but easier to compute with more data!

# Monte Carlo simulations

A

Single equilibrium in the model
One equilibrium in the data

Implementation details:

- Leapfrogging model with $N = 2$ Bertrand competitors deciding whether to invest in cost-reducing technology (IRS, 2016)
- $k_1$ parameter in investment cost function
- $M = 1000,\ T = 5$
- All methods are initialized with 2-step CCP estimator

B

Multiple equilibria in the model
Same equilibrium played the data

C

Multiple equilibria in the model
Multiple equilibria in the data:

- Long panels, each market plays their own equilibrium
- Groups of markets play the same equilibrium

# Monte Carlo A: no multiplicity

Number of equilibria at true parameter: 1
Number of equilibria in the data: 1

| | 2step | NPL | EPL | MPEC-VP | MPEC-P | NRLS |
|---|---|---|---|---|---|---|
| True $k_1 = 3.5$ | 3.52786 | 3.49714 | 3.49488 | 3.49488 | 3.49486 | 3.49488 |
| Bias | 0.02786 | -0.00286 | -0.00512 | -0.00512 | -0.00514 | -0.00512 |
| MCSD | 0.10037 | 0.06522 | 0.07042 | 0.07042 | 0.07078 | 0.07042 |
| ave log-like | -1.16661 | -1.16144 | -1.16143 | -1.16143 | -1.16139 | -1.16143 |
| log-likelihood | -5833.07 | -5807.21 | -5807.16 | -5807.16 | -5806.95 | -5807.16 |
| log-like short | - | -0.050 | -0.000 | -0.000 | -0.000 | -0.000 |
| KL divergence | 0.03254 | 0.00021 | 0.00024 | 0.00024 | 0.00024 | 0.00024 |
| $\|\|P - P0\|\|$ | 0.11270 | 0.00469 | 0.00495 | 0.00495 | 0.00500 | 0.00495 |
| $\|\|\Psi(P) - P\|\|$ | 0.16185 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\|\|\Gamma(v) - v\|\|$ | 0.87095 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| Converged of 100 | - | 100 | 100 | 100 | 99 | 100 |

- Equilibrium conditions satisfied (except 2step)
- Nearly all MLE estimators identical to the last digit
- NPL and EPL estimators approach MLE

# Monte Carlo B: discontinuous likelihood

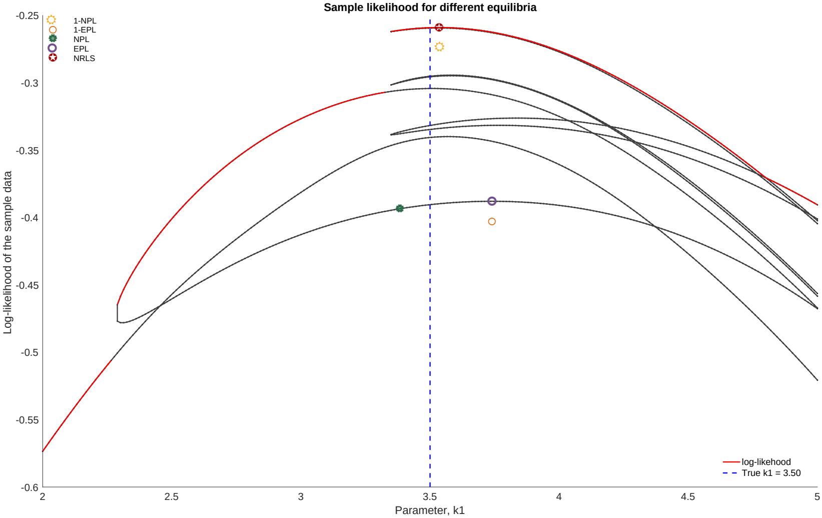
Number of equilibria at true parameter: 9
Number of equilibria in the data: 1
Data generating equilibrium: unstable, near "cliffs"

| | 2step | NPL | EPL | MPEC-VP | MPEC-P | NRLS |
|---|---|---|---|---|---|---|
| True k1=3.5 | 3.49739 | 3.55144 | 3.64772 | 3.65943 | 3.67027 | 3.50212 |
| Bias | -0.00261 | 0.05144 | 0.14772 | 0.15943 | 0.17027 | 0.00212 |
| MCSD | 0.13999 | 0.07133 | 0.12900 | 0.12693 | 0.11583 | 0.03255 |
| ave log-like | -0.27494 | -0.29474 | -0.29528 | -0.30330 | -0.30257 | -0.25086 |
| log-likelihood | -1374.721 | -1473.695 | -1476.425 | -1516.503 | -1512.847 | -1254.320 |
| log-like short | - | -219.375 | -222.104 | -270.999 | -267.523 | -0.000 |
| KL divergence | 0.01512 | 0.04889 | 0.04495 | 0.04102 | 0.04078 | 0.00016 |
| $\|\|P - P_{\mathbf{0}}\|\|$ | 0.62850 | 0.86124 | 0.83062 | 0.66562 | 0.65879 | 0.01610 |
| $\|\|\Psi(P) - P\|\|$ | 0.763764 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000002 |
| $\|\|\Gamma(v) - v\|\|$ | 0.852850 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000005 |
| N runs of 100 | 100 | 100 | 100 | 28 | 27 | 100 |

- Equilibrium conditions are satisfied, but estimators converge to *wrong* equilibria as seen from KL divergence from DGP equilibria
- Biased estimates by EPL, NPL and MPEC
(constraints are satisfied, yet low likelihood and high KL divergence)

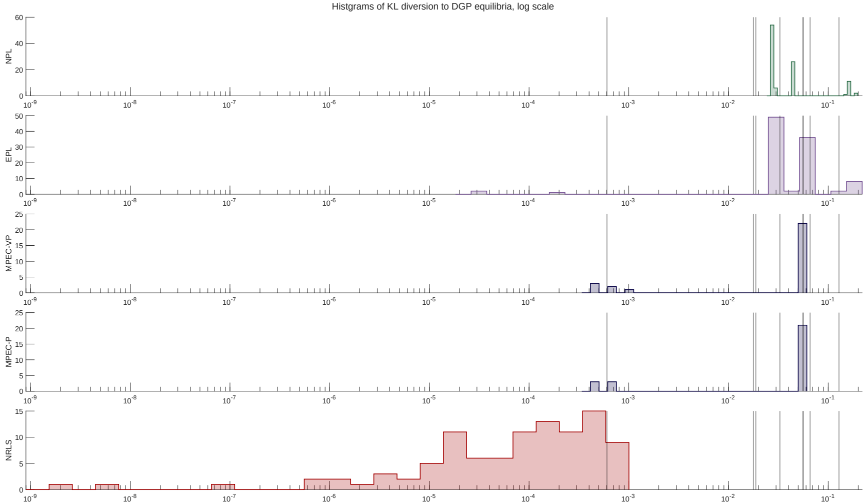
# Likelihood correspondence

## Lines are costructed using symmetric KL-divergence



# Equilibrium selection estimates

## Distribution of KL-divergence to the DGP equilibrium, vertical lines represent other equilibria



# Monte Carlo B: massive multiplicity

Number of equilibria at true parameter: 2455
Number of equilibria in the data: 1
Time to enumerate all equilibria (RLS) once: 10m 39s

| | 1-NPL | NPL | EPL | NRLS |
|---|---|---|---|---|
| True k1=3.75 | 3.70959 | 3.71272 | 3.78905 | 3.74241 |
| Bias | -0.04041 | -0.03728 | 0.03905 | -0.00759 |
| MCSD | 0.11089 | 0.06814 | 0.40716 | 0.03032 |
| ave log-likelihood | -0.38681557 | -0.37348793 | -0.45256293 | -0.35998461 |
| log-likelihood | -1934.078 | -1867.440 | -2262.815 | -1799.923 |
| log-like shortfall | - | -66.529 | -467.607 | -0.000 |
| KL divergence | Inf | 14.07523 | 12231.59186 | 0.32429 |
| $\|\|P - P_0\|\|$ | 0.82204 | 0.65580 | 0.79241 | 0.07454 |
| $\|\|\Psi(P) - P\|\|$ | 0.963574 | 0.000000 | 0.000000 | 0.000006 |
| $\|\|\Gamma(v) - v\|\|$ | 7.020899 | 0.000000 | 0.000000 | 0.000008 |
| N runs of 100 | 100 | 18 | 68 | 100 |
| CPU time | 0.159s | 11.262s | 4.013s | 4.731s |

- Severe convergence problems for NPL and EPL
- Poor eqb identification (low likelihood and high KL divergence)
- NRLS has comparable CPU time (much faster than full enumeration)

# Monte Carlo C, multiple equilibria in the data

- Assume that the same equilibrium is played in each market over time
- Grouped fixed-effects, groups defined by the equilibria played

1. Joint grouped fixed-effects estimation
   - Estimate the partition of the markets into groups playing different equilibria together with $\theta$
   - For each market compute maximum likelihood over all equilibria and "assign" it to the relevant group (estimation+classification)
   - Computationally very demanding: BnB market-by-market, non-parametric refinement has no bite
2. Two-step grouped fixed-effects estimation
   - Step 1: partition the markets based on some observable characteristics (K-means clustering) (Outside of Monte Carlo)
   - Step 2: estimate $\theta$ allowing different equilibria in different groups
   - Small additional computational cost for NRLS!

Bonhomme, Manresa (2015); Bonhomme, Lamadon, Manresa (2022)

# Monte Carlo C: multiple equilibria in the data

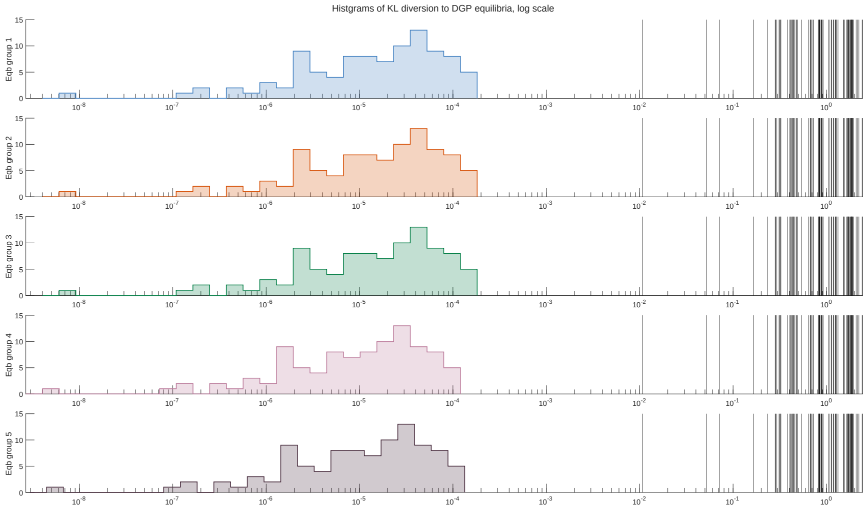
Number of equilibria at true parameter: 81
Number of equilibria in the data: 5
Number of unique equilibria in the data: 3

| | 1-NPL | NRLS | | | | |
|---|---|---|---|---|---|---|
| True k1=9.25 | 9.20991 | 9.25449 | | | | |
| Bias | -0.04009 | 0.00449 | | | | |
| MCSD | 0.15021 | 0.04109 | | | | |
| ave log-likelihood | -0.798223 | -0.707174 | | | | |
| log-likelihood | -19955.57 | -17679.36 | | | | |
| log-like shortfall | - | 0.000 | | | | |
| KL divergence | 0.32943 | 0.00039 | 0.00039 | 0.00039 | 0.00040 | 0.00028 |
| $\|\|P - P_{\mathbf{0}}\|\|$ | 0.32787 | 0.00287 | 0.00287 | 0.00287 | 0.00252 | 0.00240 |
| $\|\|\Psi(P) - P\|\|$ | 0.460870 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| $\|\|\text{Bellman}(V) - V\|\|$ | 5.438776 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| # converged of 100 | 100 | 100 | | | | |
| CPU time, sec | 0.023 | 20.695 | | | | |

- All 5 equilibria were identified correctly as seen from KL divergence
- The first three equilibria are the same in DGP, and have the same KL and L1 divergence
- Similar results in runs with many more equilibria in the data

# Equilibrium selection estimates

## Distribution of KL-divergence to the DGP equilibrium, vertical lines represent other equilibria



# NRLS estimator for directional dynamic games

Complicated computational task involving maximization over the large finite set of all MPE equilibria $\rightarrow$ branch-and-bound algorithm with combined likelihood bounding rule

1. Each stage game $\rightarrow$ non-linear solver, specific to the model
2. Combining stage game solutions to full game MPEs $\rightarrow$ State Recursion algorithm
3. Solving for all MPE equilibria $\rightarrow$ Recursive Lexicographic Search
4. Structural estimation $\rightarrow$ Nested Recursive Lexicographic Search

- Implementation of statistically efficient estimator (MLE)
- Using BnB NRLS avoids full enumeration at no cost
- BnB augmented with non-parametric likelihood bounding function $\rightarrow$ less computation with larger sample size
- Computationally trackable
- Fully robust to multiplicity of equilibria